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Nonconvex Energy Functions, Related Eigenvalue Hemivariational Inequalities on the Sphere and Applications

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Abstract. The present paper deals with a new type of eigenvalue problems arising in problems involving nonconvex nonsmooth energy functions. They lead to the search of critical points (e.g. local minima) for nonconvex nonsmooth potential functions which in turn give rise to hemivariational inequalities. For this type of variational expressions the eigenvalue problem is studied here concerning the existence and multiplicity of solutions by applying a critical point theory appropriate for nonsmooth nonconvex functionals.

Key words: Nonconvex energy, hemivariational inequalities, eigenvalue problems.

1. Introduction

Nonconvex and possibly nonsmooth energy functions in boundary value problems in Mechanics and Engineering lead to new types of variational expressions called hemivariational inequalities. Their solution characterizes the equilibrium of the problem studied in steady case problems, whereas in dynamic problems the evolution of the phenomenon. The theory of hemivariational inequalities is closely related with the search of local critical points (e.g. minima or maxima) for nonconvex, nonsmooth energy functions. These points are called substationary points. Indeed under certain very general assumptions the two problems, i.e. search for solutions of a hemivariational inequality and search for local minima or maxima of the corresponding energy are equivalent. In the case of convexity, a hemivariational inequality reduces to a variational inequality, and the search for a local critical point to the search for a global minimum.

The theory of hemivariational inequalities has been introduced and developed by the work of P. D. Panagiotopoulos in Mechanics [1]–[5] concerning the derivation of variational "principles" for problems involving nonconvex nonsmooth energy functions. We refer to [3] for all related references and to [4], [5] concerning the largest area of Nonsmooth Mechanics. Mathematical questions concerning the existence of solutions of hemivariational inequalities have been treated in [2], [3] by means of compactness arguments and by Z. Naniewicz in [6]–[8] by means of pseudomonotonicity arguments for multivalued mappings. In this respect see [9].

Here we study the corresponding eigenvalue problem. To this end we apply a critical point approach appropriate for nonconvex nonsmooth functionals. This approach extends the approach due to D. Motreanu and P. D. Panagiotopoulos [10] and to D. Motreanu and Z. Naniewicz [11].

It is well-known that the solution of an eigenvalue problem is closely connected with the stability analysis of the corresponding physical system. Having here in mind problems of elasticity or generally of engineering (adhesively connected von Kármán plates) and problems of economics (network flow problems cf. [9]) leading to hemivariational inequalities, for which either the solution remains a priori bounded, or the cost or the weight of the structure are prescribed we have studied the corresponding eigenvalue problem on a sphere

$$S_r = \{u \in V : \|u\|_V = r\}, \quad r > 0$$

in a real Hilbert space V. The proof followed here takes into account the geometry of the sphere S_r. In Section 2 the existence of the solution is investigated. Section 3 deals with the multiplicity of solutions and Section 4 describes certain applications of the method concerning constant weight or cost problems.

2. The Existence Result

Let V be a real Hilbert space, with the scalar product $(\cdot,\cdot)_V$ and the associated norm $\|\cdot\|_V$, which is densely and compactly imbedded in $L^p(\Omega;\mathbb{R}^N)$ for some $p \geqslant 2$, an integer $N \geqslant 1$ and a bounded domain Ω in $\mathbb{R}^m$, $m \geqslant 1$. The pairing over $V \times V^*$ is denoted by $\langle\cdot\rangle_V$. The Euclidean norm in any Euclidean space $\mathbb{R}^N$ is denoted by $|\cdot|$ and the pairing over $\mathbb{R}^N \times (\mathbb{R}^N)^*$ by $\langle\cdot,\cdot\rangle_{\mathbb{R}^N}$.

In the space V we consider the sphere S_r of center 0 and radius $r > 0$, i.e., $\|u\|_V = r$, regarded as a Riemannian manifold with the Riemannian structure induced by the Hilbert space V. The (geodesic) distance on S_r is denoted by $d(\cdot,\cdot)$, that is, for the points $u, v \in S_r$, $d(u,v)$ is equal to the length of the minimal arc of the great circle on S_r joining u and v.

Since V is continuously imbedded in $L^p(\Omega;\mathbb{R}^N)$, there is a constant $c_0 > 0$ such that

$$\|v\|_{L^p} \leqslant c_0\|v\|_V, \quad v \in V. \tag{1}$$

Let $\alpha : V \times V \to \mathbb{R}$ be a continuous symmetric bilinear form, let $C : S_r \to V^*$ a compact (nonlinear) mapping (in the sense that the closure of the range $C(S_r)$ of C is compact in V^*) and let $j : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ be a Carathéodory function, locally Lipschitz in the second variable and with $j(\cdot,0)$ bounded on Ω. We denote by $A : V \to V^*$ the operator which corresponds to the bilinear form α,

$$\langle Au, v\rangle_V = \alpha(u,v), \quad u, v \in V.$$

For the sake of simplicity we denote Clarke's [12] generalized gradient $\partial_y j(x,y)$ of j with respect to the second variable simply by $\partial j(x,y)$. The same convention will be used relative to Clarke's generalized directional derivative $j^0(x,y;z)$. For any other locally Lipschitz real-valued functional G, the notations ∂G and G^0 will always designate the generalized gradient and the generalized directional derivative, respectively.

This Section deals with the following eigenvalue problem with constraints: Find $u \in V$ and $\lambda \in \mathbb{R}$ such that the two relations below (2), (3) hold

$$\alpha(u,v) + \langle C(u), v\rangle_V + \int_\Omega j^0(x, u(x); v(x))\,\mathrm{d}x \geqslant \lambda(u,v)_V, \quad v \in V, \tag{2}$$

$$\|u\|_V = r. \tag{3}$$

The following hypotheses are imposed:

(H_1) j satisfies the growth condition
$$|w| \leqslant c(1+|y|^{p-1}) \quad \text{for all} \quad w \in \partial j(x,y), \quad x \in \Omega, \quad y \in \mathbb{R}^N,$$
with some constant $c > 0$.

(H_2) There exists a (Fréchet) differentiable function $g : V \to \mathbb{R}$ and a lower semicontinuous (l.s.c.) on S_r function $h : V \to \mathbb{R}$ such that
$$\langle C(u), v\rangle_V \geqslant \langle g'(u), v\rangle_V + h(u+v) - h(u) \tag{4}$$
for all $u \in S_r$ and $v \in T_u S_r$;
$$g(u) + h(u) \geqslant c_1, \quad u \in S_r, \tag{5}$$
for a constant c_1;
$$h(\exp_u(tv)) \leqslant (1-t)h(u) + th(u+v), \tag{6}$$
for $u \in S_r$, $v \in T_u S_r$ and $0 < t < 1$.

(H_3) For every sequence $\{u_n\} \subset S_r$ with $u_n \to u$ weakly in V, $\alpha(u_n,u_n) + \langle C(u_n), u_n\rangle_V \to \alpha_0 \in \mathbb{R}$, and for every $w \in L^{p/(p-1)}(\Omega;\mathbb{R}^N)$ with
$$w(x) \in \partial j(x, u(x)) \quad \text{for a.e.} \quad x \in \Omega \tag{7}$$
such that $(A - \lambda_0 \Lambda)u_n$ converges in V^*, where
$$\lambda_0 := r^{-2}\left(\alpha_0 + \int_\Omega \langle w(x), u(x)\rangle_{\mathbb{R}^N}\,\mathrm{d}x\right),$$
there exists a strongly convergent subsequence of $\{u_n\}$ in V (thus in S_r).

In the statement of hypothesis (H_2), the notation $T_u S_r$ means the tangent space of S_r at $u \in S_r$, thus

$$T_u S_r = \{v \in V : (u,u)_V = 0\}. \tag{8}$$

The notation $\exp_u$ in (6) denotes the exponential mapping (in the sense of Riemannian manifolds) of S_r at $u \in S_r$, $\exp_u : T_u S_r \to S_r$, given by

$$\exp_u(v) = \gamma_v(1), \quad v \in T_u S_r, \tag{9}$$

where the curve $\gamma_v(t)$ in (9) stands for the unique geodesic (arc of a great circle) of S_r satisfying

$$\gamma_v(0) = u \quad (\text{if } v \in T_u S_r) \quad \text{and} \quad \gamma_v'(0) = v.$$

The mapping $\Lambda : V \to V^*$ of (H_3) represents the duality mapping

$$\langle \Lambda u, v \rangle_V = (u, v)_V, \quad u, v \in V.$$

Now we are in the position to formulate our existence result for the eigenvalue problem (2), (3).

THEOREM 1. *Under hypotheses* (H_1)–(H_3) *the eigenvalue problem (2), (3) admits a solution* $(u, \lambda) \in V \times \mathbb{R} (u \in S_r)$ *with*

$$\lambda = r^{-2} \left(\alpha(u, u) + \langle C(u), u \rangle_V + \int_\Omega \langle w(x), u(x) \rangle_{\mathbb{R}^N} \, dx \right)$$

for certain $w \in L^{p/(p-1)}(\Omega; \mathbb{R}^N)$ *satisfying (7).*

Proof. By (H_1) and Lebourg's mean value theorem (see Clarke [12], p. 41) one obtains that j verifies the following growth condition

$$\begin{aligned} |j(x,y)| &\leqslant |j(x,0)| + |j(x,y) - j(x,0)| \\ &\leqslant |j(x,0)| + (\sup\{|w| : w \in \partial j(x, \bar{y}), \bar{y} \in [0, y]\})|y| \\ &\leqslant |j(x,0)| + c(|y| + |y|^p) \\ &\leqslant c_2 + c_3 |y|^p, \quad \forall x \in \Omega, \quad y \in \mathbb{R}^N, \end{aligned} \tag{10}$$

with positive constants c_2 and c_3.

We introduce the functionals $J : L^p(\Omega; \mathbb{R}^N) \to \mathbb{R}$ by setting

$$J(v) = \int_\Omega j(x, v(x)) \, dx, \quad v \in L^p(\Omega; \mathbb{R}^N), \tag{11}$$

$E : V \to \mathbb{R}$ by setting

$$E(v) = (1/2)\alpha(v, v) + g(v), \quad v \in V \tag{12}$$

and $I : V \to \mathbb{R}$ by setting

$$I = E + J|_V + h. \tag{13}$$

From (1), (5) and (10) we see that I is bounded from below on S_r

$$\begin{aligned} I(u) &\geqslant -(1/2)\|\alpha\| \, \|u\|_V^2 + g(u) + h(u) - c_2|\Omega| - c_3 \|u\|_{L^p}^p \\ &\geqslant -(1/2)\|\alpha\| r^2 + c_1 - c_2 |\Omega| - c_3 c_0^p r^p, \quad u \in S_r. \end{aligned} \tag{14}$$

By (14) and because the functional I defined in (13) is l.s.c. we can apply to I the Ekeland's variational principle on the complete metric space S_r (see Ekeland [13]). Then there exists a sequence $\{u_n\} \subset S_r$ such that

$$I(u_n) \leqslant \inf_{S_r} I + \frac{1}{n} \tag{15}$$

and

$$I(w) \geqslant I(u) - (1/n)d(w, u_n) \quad \text{for each} \quad w \in S_r. \tag{16}$$

If we set

$$w = \exp_{u_n}(tv) \quad \text{with} \quad t > 0 \quad \text{and} \quad v \in T_{u_n}S_r,$$

inequality (16) becomes

$$\begin{aligned} &E(\exp_{u_n}(tv)) - E(u_n) + J(\exp_{u_n}(tv)) - J(u_n) + h(\exp_{u_n}(tv)) - h(u_n) \\ &\quad \geqslant -(1/n)d(\exp_{u_n}(tv), u_n), \end{aligned}$$

or, in view of (6),

$$\begin{aligned} &E(\exp_{u_n}(tv)) - E(u_n) + J(\exp_{u_n}(tv)) - J(u_n) + t(h(u_n + v) - h(u_n)) \\ &\quad \geqslant -(1/n)d(\exp_{u_n}(tv), u_n), \end{aligned} \tag{17}$$

for $v \in T_{u_n}S_r$ and $t > 0$.

We recall the following properties for the exponential mapping $\exp_{u_n} : T_{u_n}S_r \to S_r$

$$\frac{d}{dt}(\exp_{u_n}(tv))|_{t=0} = v, \quad v \in T_{u_n}S_r, \tag{18}$$

$$d(\exp_{u_n}(tv), u_n) = \|tv\|_V = t\|v\|_V \tag{19}$$

for $v \in T_{u_n}S_r$ and $t > 0$ sufficiently small.

Dividing by $t > 0$ and letting $t \to 0$ in (17) we obtain that

$$\begin{aligned} &\alpha(u_n, v) + \langle g'(u_n), v\rangle_V + h(u_n + v) - h(u_n) + J^0(u_n; v) \\ &\quad \geqslant -(1/n)\|v\|_V \text{ for all } \quad v \in T_{u_n}S_r. \end{aligned} \tag{20}$$

To write (20) we made use of the differentiability of the functional E in (12), the definition of $J^0(u_n; v)$ and formulas (18), (19). From (20) and (4) we then derive that

$$\alpha(u_n, v) + \langle C(u_n), v\rangle_V + J^0(u_n; v) \geqslant -(1/n)\|v\|_V, \quad v \in T_{u_n}S_r. \tag{21}$$

Notice that the left-hand side of (21) is continuous, convex with respect to $v \in T_{u_n}S_r$ and vanishes at 0. Consequently, Lemma 1.3 in Szulkin [14] can be applied in the tangent space $T_{u_n}S_r$. It yields an element $z_n \in (T_{u_n}S_r)^*$ of norm $\leqslant 1$ verifying

$$\alpha(u_n, v) + \langle C(u_n), v\rangle_V + J^0(u_n; v) \geqslant (1/n)\langle z_n, v\rangle_V, \quad v \in T_{u_n}S_r. \tag{22}$$

Applying Hahn–Banach theorem one obtains an element (denoted again by z_n) in V^* with

$$\|z_n\|_{V^*} \leqslant 1 \tag{23}$$

so that (22) remains true (on $T_{u_n}S_r$).

The density of V in $L^p(\Omega;\mathbb{R}^N)$ implies (cf. Chang [15], Theorem 2.2)

$$\partial(J|_V)(u) \subset \partial J(u), \quad u \in V. \tag{24}$$

Then from (22) and (24) one finds an element

$$w_n \in \partial J(u_n), \quad n \geqslant 1 \tag{25}$$

satisfying the equality

$$\langle Au_n + C(u_n) + w_n - (1/n)z_n, v\rangle_V = 0, \quad v \in T_{u_n}S_r. \tag{26}$$

By (26) and the characterization of the tangent space $T_{u_n}S_r$ (see (8)) there exist numbers $\lambda_n \in \mathbb{R}$ such that

$$Au_n + C(u_n) + w_n - (1/n)z_n = \lambda_n \Lambda u_n, \quad n \geqslant 1. \tag{27}$$

The boundedness of $\{u_n\} \subset S_r$ ensures the existence of a subsequence again denoted by $\{u_n\}$ such that

$$u_n \to u \text{ weakly in } V, \tag{28}$$

$$u_n \to u \text{ strongly in } L^p(\Omega;\mathbb{R}^N), \tag{29}$$

$$\{C(u_n)\} \text{ is strongly convergent in } V^*, \tag{30}$$

$$\alpha(u_n, u_n) + \langle C(u_n), u_n\rangle_V \to \alpha_0 \in \mathbb{R}. \tag{31}$$

Here we used also the compactness of the imbedding $V \subset L^p(\Omega;\mathbb{R}^N)$ and of the mapping $C : S_r \to V^*$. Since the functional J in (11) is locally Lipschitz on $L^p(\Omega;\mathbb{R}^N)$, relations (25) and (29) imply that the sequence $\{w_n\}$ is bounded in $L^p(\Omega;\mathbb{R}^N)$. According to the compact imbedding $V^* \subset L^{p/(p-1)}(\Omega;\mathbb{R}^N)$ we may suppose that for some $w \in V^*$ one has

$$w_n \to w \text{ strongly in } V^*. \tag{32}$$

The upper semicontinuity of the generalized gradient ∂J (cf. Clarke [12], Proposition 2.1.5 (b)) and relations (29), (25) imply that

$$w \in \partial J(u). \tag{33}$$

We note that the growth condition (H_1) is just Hypothesis B in [12], p. 83, so it is allowed to apply Theorem 2.7.5 of [12]. It follows that (33) implies (7).

By scalar product multiplication of (27) with u_n one obtains

$$\alpha(u_n, u_n) + \langle C(u_n) + w_n - (1/n)z_n, u_n\rangle_V = \lambda_n r^2.$$

This equality and relations (31)–(33), (23), (28) show that

$$\lambda_n \to \lambda_0 := r^{-2}\left(\alpha_0 + \int_\Omega \langle w(x), u(x)\rangle_{\mathbb{R}^N}\,\mathrm{d}x\right), \quad \text{as } n \to \infty. \tag{34}$$

Furthermore, from (27), (30), (32) and (23) we get that

$$\{Au_n - \lambda_n \Lambda u_n\} \text{ converges in } V^*. \tag{35}$$

Combining (34), (35) we conclude that $\{(A - \lambda_0\Lambda)u_n\}$ converges in V^*. This fact together with (31), (33), (7) permit us to apply hypothesis (H_3). We arrive at the conclusion that along a subsequence

$$u_n \to u \text{ strongly in } V, \tag{36}$$

with $u \in S_r$.

Passing to the limit in (15) we see that

$$\lim_{n\to\infty} I(u_n) = \inf_{S_r} I. \tag{37}$$

The lower semicontinuity of h on S_r and relations (36), (37) enable us to write

$$\begin{aligned} h(u) \leqslant \liminf_{n\to\infty} h(u_n) &= \lim_{n\to\infty}(I(u_n) - E(u_n) - J(u_n)) \\ &= \inf_{S_r} I - E(u) - J(u), \end{aligned}$$

and therefore

$$I(u) = \inf_{S_r} I. \tag{38}$$

Due to (38) we may write that

$$\begin{aligned} 0 &\leqslant I(\exp_u(tv)) - I(u) \\ &= E(\exp_u(tv)) - E(u) + J(\exp_u(tv)) - J(u) + h(\exp_u(tv)) - h(u), \\ &\qquad v \in T_u S_r \quad \text{and} \quad t > 0, \end{aligned}$$

or, by (6),

$$\begin{aligned} &t^{-1}(E(\exp_u(tv)) - E(u)) + t^{-1}(J(\exp_u(tv)) - J(u)) + h(u+v) - h(u) \\ &\quad \geqslant 0. \end{aligned}$$

Letting $t \to 0$ in the foregoing inequality we derive by means of (18) that

$$a(u,v) + \langle C(u), v\rangle_V + J^0(u;v) \geqslant 0, \quad v \in T_u S_r. \tag{39}$$

Relation (39) can be written in the inclusion form

$$-(Au + C(u))|_{T_u S_r} \in \partial(J|_{S_r})(u).$$

This is equivalent to the existence of some $w \in V^*$ satisfying (33) and

$$\langle Au + C(u) + w, v\rangle_V = 0, \quad v \in T_u S_r. \tag{40}$$

Comparing (8) and (40) it results that $\lambda \in \mathbb{R}$ exists with the property

$$Au + C(u) + w = \lambda u. \tag{41}$$

Due to (33) and hypothesis (H_1) one has (cf. [12], p. 83)

$$w \in \partial J(u) \subset \int_\Omega \partial j(x, u(x))\,\mathrm{d}x$$

in the sense that $w \in L^{p/(p-1)}(\Omega; \mathbb{R}^N)$ and (7) holds. Then (41) and Proposition 2.1.2 of [12] imply that

$$\begin{aligned}\lambda(u, v)_V &= \alpha(u, v) + \langle C(u), v\rangle_V + \int_\Omega \langle w(x), v(x)\rangle_{\mathbb{R}^N}\,\mathrm{d}x \\ &\leqslant \alpha(u, v) + \langle C(u), v\rangle_V + \int_\Omega j^0(x, u(x); v(x))\,\mathrm{d}x, \quad v \in V,\end{aligned}$$

which is just (2). Equality (3) was already proved (see (36)). The formula determining the eigenvalue λ follows directly from (41). This completes the proof of Theorem 1.

COROLLARY 2. *Under assumptions* (H_1)–(H_3) *the variational hemivariational inequality*

$$\alpha(u, v) + \langle g'(u), v\rangle_V + h(u + v) - h(u) + \int_\Omega j^0(x, u(x); v(x))\,\mathrm{d}x \geqslant 0$$

for all $v \in V$ *with* $(u, v)_V = 0$,

has a solution $u \in V$ *with* $\|u\|_V = r$.

Proof. This result has been obtained subsequently in the proof of Theorem 1. More precisely, it is deduced from the inequality preceding (39).

EXAMPLE 3. (a) For the sake of simplicity assume that on a Hilbert space V, $\alpha : V \times V \to \mathbb{R}$ is coercive, i.e.

$$\alpha(v, v) \geqslant \bar{a}\|v\|_V^2, \quad v \in V$$

where $\bar{a}$ is a constant > 0, that $C = 0$ and that $j : \Omega \times \mathbb{R}^N \to \mathbb{R}$ verifies (H_1) and the following generalized sign condition

$$\int_\Omega \langle w(x), u(x)\rangle_{R^N}\,\mathrm{d}x < 0, \tag{42}$$

for all $u \in S_r$ and for $w \in L^{p/(p-1)}(\Omega; \mathbb{R}^N)$ satisfying (7). Then (H_2) holds with $g = h = 0$ and (H_3) is also valid because, by (42), λ_0 in (H_3) satisfies $\lambda_0 < \bar{a}$ for all

$u \in S_r$. Consequently, λ_0 is in the resolvent set of the operator $A : V \to V^*$; thus (H_3) is true. Let us observe that different other situations can be treated analogously, for example, when λ_0 is an eigenvalue of A, but with the corresponding eigenspace of a finite dimension. This is due to the a priori boundedness $\|u_n\|_V = r$ in (H_3).

As a very particular situation of (42) let us consider the case $N = 1$ and $j : \Omega \times \mathbb{R} \to \mathbb{R}$ in the form of a primitive

$$j(x,t) = \int_0^t \beta(x,s)\,\mathrm{d}s, \quad (x,t) \in \Omega \times \mathbb{R},$$

with β measurable satisfying (H_1) with β in the place of w. Then the sign condition assumed by Chang [15] in Theorem 5.5, namely

$$\underline{j}(x,t) > 0, \quad t < 0; \quad \bar{j}(x,t) < 0, \quad t > 0, \tag{43}$$

where

$$\partial j(x,t) = [\underline{j}(x,t), \bar{j}(x,t)], \quad (x,t) \in \Omega \times \mathbb{R},$$

is less general than our assumption (42) even in this particular case. Note that (43) was written with the opposite signs in $\underline{j}, \bar{j}$ in comparison with Chang [15], Theorem 5.5, because the functional J in (11) corresponds to Chang's notation with $-J$.

(b) Concerning hypothesis (H_2) the convexity condition (6) relative to the sphere S_r is in fact a fairly weak requirement for the function $h : V \to \mathbb{R}$. For instance, let us put $h \equiv \text{constant} = k_0$ on S_r and $h =$ any function $\geqslant k_0$ on $V \backslash S_r$. Then (6) is verified and h is l.s.c. on S_r.

3. Multiplicity of Solutions. A Special Case

In this Section we investigate, under certain additional symmetry assumptions the multiplicity of solutions $(u, \lambda) \in V \times \mathbb{R}$, with $\|u\|_V = r$, for the following eigenvalue hemivariational inequality on the sphere S_r, related to the problem of the previous Section:

Find $u \in V$ and $\lambda \in \mathbb{R}$ such that

$$\begin{cases} \alpha(u,v) + C(u,v) + \displaystyle\int_\Omega j^0(x,u(x);v(x))\,\mathrm{d}x \geqslant \lambda(u,v)_V, \quad v \in V, & (2') \\ \|u\|_V = r. & (3) \end{cases}$$

The data α and j in $(2')$ have the same meaning as in Section 1, but here C stands for a real function $C : S_r \times V \to \mathbb{R}$. A weak kind of compactness assumption for C is hidden in the hypothesis below. No continuity assumption is necessary.

(H_2') There exists a locally Lipschitz function $f : V \to \mathbb{R}$, bounded on S_r, satisfying

$$C(u, v) \geqslant f^0(u; v) \text{ for all } u \in S_r \text{ and } v \in V \text{ with } (u, v)_V = 0$$

and

$$\{z \in V^* : z \in \partial f(u), \quad u \in S_r\}$$

is relatively compact in V^*.

Hypothesis (H_3) is replaced by

(H_3') for every sequence $\{u_n\} \subset S_r$ with $u_n \to u$ weakly in V, for every $z_n \in \partial f(u_n)$ with

$$\alpha(u_n, u_n) + \langle z_n, u_n \rangle_V \to \alpha_0 \in \mathbb{R},$$

where α_0 is some real number, and for every $w \in L^{p/(p-1)}(\Omega; \mathbb{R}^N)$ verifying (7) such that $(A - \lambda_0 \Lambda)u_n$ converges in V^*, for λ_0 as in (H_3), there exists a (strongly) convergent subsequence of $\{u_n\}$ in V (thus in S_r).

In addition, we suppose a symmetry condition.

(H_4) j is even with respect to the second variable $y \in \mathbb{R}^N$, i.e.,

$$j(x, y) = j(x, -y) \text{ for all } x \in \Omega, y \in \mathbb{R}^N;$$

f is even on the sphere S_r, i.e.,

$$f(u) = f(-u) \text{ for all } u \in V \text{ with } \|u\|_V = r.$$

Our multiplicity result concerning the eigenvalue problem (2′), (3) is formulated as follows.

THEOREM 4. *Assume that hypotheses (H_1), (H_2'), (H_3'), (H_4) are fulfilled. Then the constrained eigenvalue problem (2′), (3) admits infinitely many pairs of solutions* $\{(\pm u_n, \lambda_n)\}_{n \geqslant 1} \subset S_r \times \mathbb{R}$ *with*

$$\lambda_n = r^{-2} \left(\alpha(u_n, u_n) + \langle z_n, v_n \rangle_V + \int_\Omega \langle w_n(x), u_n(x) \rangle_{\mathbb{R}^N} \, \mathrm{d}x \right)$$

for some $z_n \in V^*$ *and* $w_n \in L^{p/(p-1)}(\Omega; \mathbb{R}^N)$ *satisfying*

$$z_n \in \partial f(\pm u_n), \quad n \geqslant 1 \tag{44}$$

and

$$w_n(x) \in \partial j(x, \pm u_n(x)) \quad \textit{for a.e. } x \in \Omega, n \geqslant 1. \tag{45}$$

Proof. Consider the locally Lipschitz functional $F : V \to \mathbb{R}$ given by

$$F(u) = (1/2)\alpha(u, u) + f(u) + J(u), \quad u \in V, \tag{46}$$

with $J : L^p(\Omega; \mathbb{R}^N) \to \mathbb{R}$ described by (11). From (H_4) it follows that F is even on S_r

$$F(u) = F(-u) \quad \text{for } u \in V \quad \text{with } \|u\|_V = r. \tag{47}$$

As in (14) we check that F is bounded below on S_r

$$F(u) \geqslant C_0, \quad u \in S_r \tag{48}$$

for some constant C_0.

Let us now show that the Palais–Smale condition holds on S_r (in the sense of Chang [15]). To this end we recall that the generalized gradient $\partial(F|_{S_r})(u)$ at $u \in S_r$ is expressed by

$$\partial(F|_{S_r})(u) = \{J - r^{-2}\langle J, u\rangle_V \Lambda u; J \in \partial F(u)\}. \tag{49}$$

According to (49) it suffices to show that if $\{v_n\} \subset S_r$ is a sequence for which there exists a sequence $\{J_n\}$ in V^* with

$$J_n \in \partial F(v_n) \tag{50}$$

and

$$J_n - r^2\langle J_n, v_n\rangle_V \Lambda v_n \to 0 \quad \text{strongly in } V^*, \tag{51}$$

then $\{v_n\}$ contains a strongly convergent subsequence in V.

From (50), (51) and (46) and the formula for the generalized gradient of a sum ([12], Proposition 2.3.3) one deduces that two sequences $\{z_n\}$ and $\{w_n\}$ in V^* can be determined such that

$$z_n \in \partial f(v_n) \tag{52}$$

$$w_n \in \partial(J|_V)(v_n), \tag{53}$$

$$Av_n + z_n + w_n - r^2\langle Av_n + z_n + w_n, v_n\rangle_V \Lambda v_n \to 0. \tag{54}$$

Since $\|v_n\|_V = r$ we can extract a subsequence again denoted by $\{v_n\}$ such that

$$v_n \to u \quad \text{weakly in } V, \tag{55}$$

for some $u \in V$. As in the proof of Theorem 1, we can choose subsequences of $\{z_n\}$ (cf. (H_3')) and of $\{w_n\}$ for which one has

$$z_n \to z \quad \text{strongly in } V^*, \tag{56}$$

$$w_n \to w \quad \text{strongly in } V^*, \tag{57}$$

with $z, w \in V^*$. Additionally, we can suppose that

$$\{\alpha(v_n, v_n)\} \text{ is convergent in } \mathbb{R} \tag{58}$$

$$\langle z_n + w_n, v_n\rangle_V \to \langle z + w, u\rangle_V. \tag{59}$$

Due to the upper semicontinuity property of the generalized gradient (cf. [12], Proposition 2.1.5) relations (52)–(54), (59) and hypothesis (H_2') yield

$$z \in \partial f(u), \tag{60}$$

$$w \in \partial(J|_V)(u). \tag{61}$$

From (61) we deduce as in the proof of Theorem 1 that w satisfies (7). Relation (54)–(59) allow us to derive that

$$(A - \lambda_0 \Lambda)v_n \text{ converges strongly in } V^*, \tag{62}$$

with λ_0 obtained from (58), (59) as required in (H_3'). Now we apply hypothesis (H_3') with v_n in place of u_n. This justifies that a strongly convergent subsequence of $\{v_n\}$ exists and thus the Palais–Smale condition for the function F on S_r is verified.

For any closed, symmetric with respect to the origin, subset S of S_r, let us denote by $\gamma(S)$ the Krosnoselski's genus of S. Namely, $\gamma(S)$ is the smallest integer $k \geqslant 0$ for which there exists an odd continuous mapping from S into $\mathbb{R}^k \backslash \{0\}$ (see Rabinowitz [16]) for more details). We consider the following class of subsets of the sphere S_r

$$\Gamma_n = \{S \subset S_r : S \text{ closed, symmetric with respect to } 0, \\ \text{with } \gamma(S) \geqslant n\}, \quad n \geqslant 1 \tag{63}$$

and we form the corresponding minimax value of F over Γ_n

$$\beta_n = \inf_{S \in \Gamma_n} \max_{u \in S} F(u), \quad n \geqslant 1. \tag{64}$$

It is clear that each class Γ_n contains compact sets, for instance $S_r \cap V_{n+1}$ with V_{n+1} and $(n+1)$-dimensional linear subspace of V. In view of (48) it follows then that each β_n is a real number.

The Palais–Smale condition and property (48) are the only requirements which are necessary to apply Theorem 3.2 of Chang [15]. This ensures that $\beta_n \in \mathbb{R}$ given by (63), (64) are critical values of F on S_r. Hence there exists a critical point u_n (in fact $\pm u_n$ by (47)) of F, i.e.,

$$0 \in \partial F(\pm u_n) \tag{65}$$

with $F(\pm u_n) = \beta_n$. Recalling now (49) we can express (65) in the form

$$\alpha(\pm u_n, v) + \langle z_n + w_n, v\rangle_V = \lambda_n(\pm u_n, v)_V, \quad v \in V, \tag{66}$$

for $\lambda \in \mathbb{R}$ and for z_n, w_n satisfying (60), (61) where u, z, w are replaced by $\pm u_n, z_n, w_n$. Arguing as in the final part of Theorem 1 we are led to the inequality

$$\alpha(\pm u_n, v) + \langle z_n, v\rangle_V + \int_\Omega j^0(x, \pm u_n; v(x))\,dx \geqslant \lambda_n(\pm u_n, v)_V, \\ v \in V, \quad n \geqslant 1. \tag{67}$$

Then hypothesis (H_2') ensures that ($2'$) holds for $u = \pm u_n$, $n \geqslant 1$, with λ_n obtained in a straightforward way from (66). The proof is thus complete.

Remark 5. Hypotheses (H_1), (H_4) are clearly necessary for this approach. Hypothesis (H_3') is of the same nature as (H_3) and it was discussed in Example 3. Hypothesis (H_2') is verified, for instance, for f being the restriction to V of a locally Lipschitz function on $L^p(\Omega)$.

4. Applications

In several problems in Mechanics and Engineering the cost or the weight of the structure is expressed as a linear function of the norm of the unknown function. Thus the constraint $||u||_V = r$ imposed means that we have a system with prescribed cost or weight, or in some cases energy consumption. The stability analysis of such a system involving nonconvex nonsmooth potential functions (called also nonconvex superpotentials [1]–[5]) leads to the treatment of an eigenvalue problem for a hemivariational inequality on a sphere of given radius.

(i) Nonconvex semipermeability problems. In [17] P. D. Panagiotopoulos has shown that semipermeable membranes with nonmonotone possibly multivalued "flux-force" laws in Onsager's terminology holding on $\Omega' \subset \Omega \subset \mathbb{R}^3$ lead to hemivariational inequalities of the type (2) with $C = 0$. Form $\alpha(\cdot, \cdot)$ results from the Laplace operator, $u = 0$ on the boundary, $V = \overset{\circ}{H}{}^1(\Omega)$, $p = 5$, for instance, for $\Omega \subset \mathbb{R}^3$ (Sobolev compact imbedding), and $\alpha(\cdot, \cdot)$ is coercive. According to Korn's inequality for V we may consider that V is endowed with the norm $\alpha(u, u)^{1/2}$. Then condition (3) becomes a constant energy condition which may represent the cost of the semipermeable membrane. The semipermeability condition is assumed in [7] to be derived by a superpotential j in the form of a primitive of a function $\beta \in L^\infty_{loc}(R)$ satisfying (43). Thus the eigenvalue problem (2), (3) may be formulated. It arises when someone wants to investigate the stability of the membrane due to physical instabilizing effects (semipermeability conditions) under the condition of given cost. According to Example 3 Theorem 1 holds concerning the existence of solution. If moreover, the symmetry condition (H_4') holds Theorem 4 applies concerning the multiplicity of solutions. We refer further in this context to Example 3 in Section 2.

(ii) It is analogous the case of nonlinear network problems of given cost. It is shown in [9] Section 5.5.5 that this problem leads to a discrete hemivariational inequality having an eigenvalue problem on a sphere of the type (2), (3) with $C = 0$. We proceed as in the previous application and we apply Theorems 1 and 4.

(iii) The general case of $C \neq 0$ we have treated in Section 2 corresponds to the following mechanical problem. We consider adhesively connected von Kármán plates Ω_i, $i = 1, \cdots, \rho$, where the adhesive bond between them is expressed by means of a superpotential relation of the type (7). The plates are fixed along their boundaries Γ_i. It has been shown in [18], [19] that the equilibrium state of such a structure is governed by a hemivariational inequality whose left hand side coincides with the left hand side of (2) and in the right hand side the term (f, v) appears expressing the work of the given loading f. The corresponding eigenvalue problem on the sphere has the form (2), (3) for $V = V_1 \times \cdots \times V_\rho$, $V_i = \{v_i \in H^2(\Omega) | v_i = 0$ on $\Gamma_i\}$, where ρ denotes the number of adhesively connected plates and in the right hand side the term $\lambda(Bu, v)_V$ appears, where B is a linear compact symmetric operator. For the exact expressions of $\alpha(\cdot, \cdot)$, $C(\cdot)$ and B we refer to [1] p. 228. Due to the presence of B the proofs of Theorems 1 and 2 need a minor modification. Note that the existence results given in Sections 2 and 3 do not need the verification of any coercivity and pseudomonotonicity property as in [1], if (H_1), (H_2), (H_3) are verified. We can assume that the adhesive forces fulfill (H_1) and we can easily verify hypothesis (H_2) with $h = 0$ and with (4) holding as an equality, where $g(\cdot) = \frac{1}{2}R(G(\cdot), G(\cdot))$ (cf. [1], p. 255). Moreover (H_3) is also fulfilled, where p results from the compact imbedding $H^2(\Omega) \subset L^\infty(\Omega)$ (see in this context also Example 3).

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